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Global Optimization from Concave Minimization to Concave Mixed Variational Inequality

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This paper is dedicated to Professor Hoang Tuy on the occasion of his 90th birthday

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Abstract

We use techniques from global optimization to develop an algorithm for finding a global solution of nonconvex mixed variational inequality problems involving separable DC cost functions. In contrast to the convex mixed variational inequality, in these problems, a local solution may not be a global one. The proposed algorithm uses the convex envelope of the separable cost function over boxes to approximate a DC cost problem with a convex cost one that can be solved by available methods. To obtain better approximate solutions, the algorithm uses an adaptive rectangular bisection which is performed only in the space of concave variables. The algorithm is applied to solve the Nash-Cournot and Bertrand equilibrium models with logarithm and quadratic concave costs. Computational results on a lot number of randomly generated data show that the proposed algorithm is efficient for these models, when the number of the concave cost functions is moderate, while the size of the model may be much larger.

Keywords DC optimization · Nonconvex mixed variational inequality · Nash-Cournot oligopolistic model · Concave cost · Global solution · Gap function · Convex envelope

Mathematics Subject Classification (2010) 47J25 · 47N10 · 90C25

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1 Introduction

Nowadays, a global optimization problem is understood as the problem of finding a globally optimal solution of a mathematical programming problem, where a local optimal solution may not be a global one. The pioneer paper on global optimization was published in 1964 by Tuy [28]. In this paper, the problem of globally minimizing a concave function under linear constraints was first considered and a cutting plane method was developed for this problem. Since then, some classes of global optimization such as the DC [29], reverse convex [9], bilinear [11], convex-concave [15–17], and monotonic [30] programming problems have been considered, and some solution approaches have been developed to these classes of global optimization problems. For more detail, we refer the readers to the monographs [10, 23, 31].

Inspired and motivated by Tuy's works [28, 29], in this paper, we consider a mixed variational inequality (shortly MVI), where the cost function is DC (the difference of two convex functions) rather convex. Such a problem arises from real-life Nash-Cournot and Bertrand models involving concave cost. The concavity property of the cost function means that the cost for the production of a unit commodity does decrease as the amount of the production gets larger. Because of the concavity of the cost function, a local solution of such a MVI may not be a global one (see an example at the beginning of Section 3).

Precisely, the problem to be considered in this paper is formulated as a mixed variational inequality of the form

$$\text{Find } x^* \in D : \langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, \forall x \in D, \tag{P}$$

where D is a bounded, closed convex set in $\mathcal{R}^N$, $F : \mathcal{R}^N \to \mathcal{R}^N$ with $\text{dom}\, F \supseteq D$ and $\varphi : \mathcal{R}^N \to \mathcal{R}$ (called the cost function) is a DC function. Clearly, in a special case, when $F \equiv 0$, Problem (P) is reduced to a DC minimization one [22]. In what follows, we call a MVI where φ is DC (concave) a *DC (concave) mixed variational inequality*, whereas a MVI with φ being convex a *convex mixed variational inequality*. Such a solution x^* of Problem (P) is called a global one in contrast to a local solution that is defined as a vector $x^* \in D$ such that $\langle F(x^*), x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, \forall x \in D \cap U$, where U is a neighborhood of x^*. Clearly, every global solution is a local one, but, because of nonconvexity of φ, a local solution may not be a global one. It is worth mentioning that a Nash-Cournot oligopolistic equilibrium model where the cost function is concave was considered in [18, 19, 24] is a special case of Problem (P). In [19, 24], proximal point and splitting algorithms were proposed for finding a stationary point of this model, whereas a search-and-check algorithm was developed in [18] for finding a global solution of the model with piecewise linear concave cost function. Recently, a differentiated Nash-Cournot oligopolistic equilibrium model with concave cost function was considered in [2]. A vector optimization approach was also proposed in [25] to a Nash-Cournot oligopolistic equilibrium model with concave cost, and a branch-and-bound algorithm for finding a Pareto optimal solution of the model was developed there.

In this paper, we propose an algorithm for finding a global solution of concave mixed variational inequality problem (P) by employing the techniques commonly used in global optimization problem such as the convex envelope of a concave function and rectangular subdivision to approximate the concave mixed variational inequality (P) by convex ones. The latter then can be solved by available methods of variational inequality (see, e.g., [4, 21]).

The remaining part of the paper is organized as follows. In the next section, we describe an algorithm for approximating a global solution of Problem (P) and prove its convergence results. Section 3 is devoted to present two practical models that can be formulated in the form of DC and concave mixed variational inequality problems of form (P). We close the paper with some computational results and experiences.

2 An Algorithm for Global Solution

In this section, we describe an algorithm for approximating a global solution of DC mixed variational inequality problem (P). The idea of the proposed algorithm is quite natural; it uses the convex envelope of the cost function on each subbox to approximate the original problem with the one, where on each subbox the cost is convex. The latter then can be solved by available algorithms to obtain an approximate solution. Then by evaluating the gap function, one can check whether or not the obtained point is an ϵ-solution. If not, it uses an adaptive rectangular bisection to get a better approximate point. When the feasible set D is a box and the cost function is separable (often in practical models), the proposed algorithm can be implemented with a reasonable effort.

2.1 The Gap Function as a Stopping Criterion

Gap functions are commonly used to determine stopping rules in algorithms for optimization, variational inequality, and equilibrium problems as well as to reformulate them in the form of a mathematical programming problem [3, 6, 7]. Following this idea, we now define a gap function for the mixed variational inequality problem (P) by taking for each $x \in D$

$$g_D(x) := -\min\{\langle F(x), y - x\rangle + \varphi(y) - \varphi(x) \ : \ y \in D\}. \tag{1}$$

Clearly, $g_D(x) \geq 0$ for every $x \in D$. It is well known that $x \in D$, $g_D(x) = 0$ if and only if x is a global solution to (P) [8]. Note that when D is a box and φ is continuous, separable (often in practical models), evaluating $g_D(x)$ amounts minimizing n-continuously DC functions of one-variable on an interval.

2.2 A Search-Check-Branch Algorithm

First we recall [10, 26] that the convex envelope of a function φ on a convex set C is the convex function on C, denoted by $co_C\varphi$ such that $co_C\varphi(x) \leq \varphi(x)$ for every $x \in C$, and if ζ is any convex function on C satisfying $\zeta(x) \leq \varphi(x)$ for every $x \in C$, then $\zeta(x) \leq co_C\varphi(x)$ for every $x \in C$. It is well known [10] that the convex envelope of a concave function is affine, and that if $C = C_1 \times \cdots \times C_N$ is compact and φ is separable, i.e., $\varphi(x_1, \ldots, x_N) = \sum_{j=1}^{N} \varphi_j(x_j)$ then $co_C\varphi(x) = \sum_{j=1}^{N} co_{C_j}\varphi_j(x_j)$ where $co_{C_j}\varphi_j$ is the convex envelope of φ_j over C_j. Clearly, if φ_j is convex on the convex set C_j, then $\varphi_j \equiv co\varphi_j$ on C_j.

The algorithm we are going to describe is a search-check-branch procedure that can be outlined as follows. For a given tolerance $\epsilon \geq 0$, at each iteration, the algorithm consists of three steps. The search-step requires solving convex variational inequality problems on subboxes of D to obtain an approximate solution. The check-step uses the gap function defined above to check whether the obtained solution is an ϵ- solution or not yet. The branch-step employs an adaptive rectangular bisection performed in the space of concave variables to obtain a better approximate solution to the problem.

To be precise, suppose that the strategy set $D := D_1 \times \cdots \times D_N$ ($D_j \subset \mathcal{R}, \forall j$) and the cost function φ is separable of the form $\varphi(x) := \sum_{j=1}^{N} \varphi_j(x_j)$. Suppose further that φ_j is DC for every $j = 1, \ldots, n$ with $n \leq N$ and φ_j is convex if $j > n$. The latter condition is motivated from the fact that in some practical models the cost consists of several types, some of them are concave while the others are convex. Let

$$I^0 := D_1 \times \cdots \times D_n, \;\; J^0 := D_{n+1} \times \cdots \times D_N.$$

For a n-dimensional subbox $I \subseteq I^0$, define

$$D_I := \{x \in D \,:\, (x_1, \ldots, x_n) \in I, (x_{n+1}, \ldots, x_N) \in J^0\} \tag{2}$$

and consider the convex mixed variational inequality CMV(D_I) defined as

$$\begin{aligned}&\text{Find } x^{D_I} \in D_I \text{ such that}\\ &\langle F(x^{D_I}), y - x^{D_I}\rangle + co_I\varphi(y^I) + \sum_{j=n+1}^{N} \varphi_j(y_j) - co_I\varphi(x^I) - \sum_{j=n+1}^{N} \varphi_j(x_j) \geq 0, \forall y \in D_I,\end{aligned} \tag{3}$$

where $y^I = (y_1, \ldots, y_n)^T \in I$ is obtained by taking the first n-components of the vector $y = (y_1, \ldots, y_n, y_{n+1}, \ldots, y_N)^T$. Since $co_I\varphi(y^I) + \sum_{j=n+1}^{N} \varphi_j(y_j)$ is convex, this problem is a convex mixed variational inequality that can be solved by available methods, e.g., [4, 12, 20]. In a special case of the Nash-Cournot model presented in [18, 19], (3) is reduced to a strongly convex quadratic program.

Suppose that each strategy set D_j ($j = 1, \ldots, n$) has been divided into intervals $D_{j,1}, \ldots, D_{j,k_j}$ on each of them the cost function is convex. Let Δ be the set of n-dimensional subboxes defined as

$$\Delta := \{B := I_1 \times \cdots \times I_n : I_j \in \{D_{j,1}, \ldots, D_{j,k_j}\}, j = 1, \ldots, n\}.$$

Define Σ as the family of N-dimensional subboxes by taking

$$\Sigma := \{I = B \times J^0 \,:\, B \in \Delta\}.$$

Let us define the gap function for the approximated problem by taking

$$\bar{g}(x) := -\min_{y \in D} \bar{\phi}(x, y) \tag{4}$$

with

$$\bar{\phi}(x, y) := \langle F(x), y - x\rangle + \bar{\varphi}(y) - \bar{\varphi}(x), \tag{5}$$

where $\bar{\varphi}$ is obtained from φ by taking its convex envelope on each element of Σ.

Note that, since φ_i is convex on D_i for every $i = n+1, \ldots, N$, the convex envelope of φ_i on any subbox coincides with φ_i for every $i = n+1, \ldots, N$. In particular, $co_D\varphi$ is convex and

$$co_D\varphi(x) = \sum_{j=1}^{n} co_{D_j}\varphi_j(x_j) + \sum_{i=n+1}^{N} \varphi_i(x_i).$$

First, we briefly describe the algorithm in [18] as follows.

Algorithm 1 (Search-and-Check). Choose a tolerance $\epsilon \geq 0$.
Step 1: Select a subbox $I \in \Sigma$.
Step 2: Solve the convex mixed variational inequality (3) to obtain a solution x^{D_I}.
Step 3:
a) If $\bar{g}(x^{D_I}) \leq \epsilon$, terminate: x^{D_I} is an ϵ- solution for the piecewise concave cost problem.
b) If $\bar{g}(x^{D_I}) > \epsilon$ and $\Sigma = \emptyset$, then terminate: the problem has no solution. Otherwise, replace Σ by $\Sigma \setminus \{I\}$ and return to Step 1.

It is obvious that in the worst case, the algorithm searches all subboxes in Σ; however, the computational results reported in [18] show that by using the gap function, in general, the algorithm finds an ϵ- solution without searching all elements of Σ.

An adaptive rectangular bisection (Rule 1). Let I be a given n-dimensional subbox of $D_1 \times \cdots \times D_n$. For $x^I \in I$, define

$$j_{\max}(I) := \text{argmax}_{1 \le j \le n} \{\varphi_j(x_j^I) - co\varphi_j(x_j^I)\}.$$

Then, we bisect I into two boxes via the middle point of edge $j_{\max}(I)$. We call this middle point the *bisection point* and $j_{\max}(I)$ the *bisection index*.

For this bisection, we have the following lemma whose proof can be found, e.g., in [15, 16].

Lemma 1 *Let $\{I^k\}$ be an infinite sequence of boxes generated by the adaptive rectangular bisection Rule 1 such that $I^{k+1} \subset I^k$ for every k. Let b^k be the bisection point and j_k be the bisection index for I^k. Then $\lim_{k\to\infty}(\varphi_{j_k}(b^k) - co_{I^k}\varphi_{j_k}(b^k)) = 0$. Consequently, $\{I_{j_k}\}$ tends to a singleton, provided that φ_{j_k} is concave, but not affine on I_{j_k} for every j_k.*

For each subbox I having n-edges I_j $(j = 1, \ldots, n)$, we define

$$\rho(I_j) := \max_{t \in I_j} \{\varphi_j(t) - co\varphi_j(t)\}$$

and

$$\rho(I) := \max\{\rho(I_j) : j = 1, \ldots, n\}. \tag{6}$$

The algorithm now can be described as follows:

Algorithm 2 (Search-Check-Branch for global solution)
Initial step. Choose a tolerance $\epsilon \ge 0$, take the initial box $I^0 := D_1 \times \cdots \times D_n$. Solve the convex mixed variational inequality CMV(D) defined as

Find

$$x^0 \in D: \ \langle F(x^0), y - x^0 \rangle + co_D\varphi(y) - co_D\varphi(x^0) \ge 0, \forall y \in D. \tag{CMV(D)}$$

Let $\Sigma_0 := \{I^0\}$.

Iteration k $(k = 0, 1, \ldots)$

At the beginning of each iteration k we have:

- Σ_k: a finite family of n-dimensional subboxes of I^0;
- $x^k \in D$: the currently best feasible point, i.e., $g(x^k)$ is smallest among the obtained feasible points so far.

Step 1.

a) If $g(x^k) \le \epsilon$, terminate: x^k is an ϵ- solution of the original problem.

b) If $g(x^k) > \epsilon$, choose $I^k \in \Sigma_k$ such that

$$\rho_k := \rho(I^k) = \max\{\rho(I); \ I \in \Sigma_k\}.$$

Step 2. Use the bisection Rule 1 described above to bisect I^k into two boxes I^{k^+} and I^{k^-}. Let j_k be the bisection index for I^k.

Step 3. Solve convex variational inequality problem CMV(D_I) with $I = I^{k^-}$ and $I = I^{k^+}$ to obtain x^{k+} and x^{k-} respectively.

Step 4. If either $g(x^{k+}) \le \epsilon$ or $g(x^{k-}) \le \epsilon$, terminate.

Otherwise, update x^k, Σ_k by taking respectively

$$x^{k+1} \in \{x^k, x^{k+}, x^{k-}\} \text{ such that } g(x^{k+1}) = \min\{g(x^k), g(x^{k-}), g(x^{k+})\},$$
$$\Sigma_{k+1} = (\Sigma_k \setminus \{I^k\}) \cup \{I^{k^-}, I^{k^+}\}.$$

Step 5. Compute the convex envelope of function φ_{j_k} on the edge j_k of the subboxes I^{k^-}, I^{k^+}, thereby to obtain the two new variational inequality problems CMV($D_{I^{k+}}$) and CMV($D_{I^{k-}}$) defined as

$$x \in D_{I^{k+}} : \langle F(x), y - x\rangle + co_{k+1}\varphi_+(y) - co_{k+1}\varphi_+(x) \geq 0, \forall y \in D_{I^{k+}}$$

and

$$x \in D_{I^{k-}} : \langle F(x), y - x\rangle + co_{k+1}\varphi_-(y) - co_{k+1}\varphi_-(x) \geq 0, \forall y \in D_{I^{k-}},$$

where $co_{k+1}\varphi_+$ and $co_{k+1}\varphi_-$ are the convex envelope of φ obtained by replacing the convex envelope of φ_{j_k} on the edge j_k of I^k with the convex envelope of φ_{j_k} on the edge j_k of I^{k^+} and I^{k^-} respectively. Increase k by one and go to Step 1 of iteration k.

We have the following convergence result.

Theorem 1 *Suppose that, at each iteration, the mixed variational inequality problem obtained by replacing the cost function by its convex envelope has a solution. Then,*
(i) *If the algorithm terminates at iteration k, then x^k is an ϵ- global solution.*
(ii) *If the algorithm does not terminate, it generates an infinite sequence $\{x^k\}$ such that any its cluster point is a global solution of the original mixed variational inequality* (P). *Furthermore $g(x^k) \searrow 0$ as $k \to \infty$.*

Proof The statement (i) is obvious.
(ii) We suppose that the algorithm never terminates. Let x^* be any cluster point of $\{x^k\}$. Then, there exists a subsequence of $\{x^{k_q}\}$ that tends to x^*. Thus, the corresponding sequence of selected intervals has a nested sequence, which, by taking a subsequence if necessary, we denote also by I^{k_q}. Since I^{k_q} is the box to be bisected at iteration k_q, by Lemma 1, $\{I^{k_q}\}$ tends to a singleton, which implies that $\varphi_{j_q}(x_{j_q}) - co\varphi_{j_q}(x_{j_q}) \to 0$ as $q \to \infty$ (j_q is the bisection index at iteration k_q). Then, by the rule for selecting the bisection index, we have $\varphi_j(x_j) - co\varphi_j(x_j) \to 0$ for every j. Let u^{k_q} be a solution of the problem obtained by replacing the cost function with its convex envelope at iteration k_q on each generated box. Then, we have $\bar{g}_{k_q}(u^{k_q}) = 0$ for every q, where $\bar{g}_{k_q}$ is the gap function for the approximated problem at iteration k_q. By the definition of the gap function g for the original problem and of $\bar{g}$ for the approximated one, and the rule for selecting bisection index, we can write

$$\bar{g}(u^{k_q}) - 2\rho_{k_q} \leq g(u^{k_q}) \leq \bar{g}(u^{k_q}) + 2\rho_{k_q}, \forall q.$$

By talking a subsequence, if necessary, we can assume that $u^{k_q} \to u^*$. Letting $q \to \infty$, since $\rho_{k_q} \to 0$, by continuity of g, we obtain $g(u^*) = 0$,which implies that u^* is a solution of the original problem.

On the other hand, since x^{k_q} is the currently best feasible point obtained at iteration k_q, we have $0 \leq g(x^{k_q}) \leq g(u^{k_q})$. Letting $q \to \infty$, by continuity of g, we obtain $0 \leq g(x^*) = g(u^*) = 0$, which means that x^* is a solution of the problem. Note that, since x^k is the currently best feasible point obtained at iteration k, by definition, the sequence $\{g(x^k)\}$ is nonincreasing. Since the whole sequence $\{x^k\}$ is bounded, it has a subsequence $\{x^{k_j}\}$

converging to some $\bar{x}$. Then, as we just have shown, $\bar{x}$ is a solution which implies $g(\bar{x}) = 0$. Then, the whole sequence $\{g(x^k)\}$ tends to 0 as well. □

For Algorithm 2 and its convergence, we have the following remark that may arise questions about solution existence as well as error bound for nonconvex mixed variational inequality problem (P).

Remark 1 (i) In Theorem 1, for convergence of the algorithm, we have required that, at each iteration, the mixed variational inequality problem obtained by replacing the cost function with its convex envelope has a solution. To our best knowledge, no sufficient condition for the existence of a solution of such a nonconvex mixed variational inequality problem is yet known. Note that in contrast to convex mixed variational inequality problems, a nonconvex one may have no solution even when the feasible domain is compact and both the cost operator and the cost function are continuous (a simple example for this case can be found in [18]).
(ii) Note that although a feasible point is a global solution of Problem (P), if its value of the gap function is zero, an ϵ-solution might be far from the actual solution. In order to avoid this situation, another stopping criterion by using, for example, an error bound would be used. Some interesting error bounds for the convex case can be found in [2], but for Problem (P), we do not know any error bound result in the literature.

3 Practical Models and Computational Experience

To begin this section, first, let us consider the following example showing that the mixed variational inequality problem (P) may have local solutions which are not global ones.

Take the problem

$$\text{Find } x^* \in D : \langle Ax^* - \alpha, y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0, \forall y \in D, \tag{7}$$

where $D = [0, 400] \times [0, 300]$,

$$A = \begin{pmatrix} 0.004 & 0.002 \\ 0.002 & 0.004 \end{pmatrix}; \ \alpha = (9.3787, 8.6865)^T; \ \varphi(x) = \varphi_1(x_1) + \varphi_2(x_2),$$

with

$$\varphi_1(x_1) = \begin{cases} 8x_1 & \text{if } 0 \leq x_1 \leq 200 \\ 5x_1 + 600 & \text{if } 200 < x_1 \leq 400. \end{cases}$$

$$\varphi_2(x_2) = \begin{cases} 5x_2 & \text{if } 0 \leq x_2 \leq 100 \\ 4x_2 + 100 & \text{if } 100 < x_2 \leq 300. \end{cases}$$

Clearly, φ is piecewise affine concave on D. Let $D_* = [0, 200] \times [100, 300]$ be a subbox of D and consider (7) restricted on this subbox. Since on D_* the function φ is affine and A is symmetric positive definite, this problem can be equivalently formulated as the strongly convex quadratic program on D_*

$$\min_{x \in D_*} \left\{ \frac{1}{2} x^T Ax + (\mu - \alpha)^T x \right\},$$

where $\mu^T = (8, 4)$. Solving this strongly convex quadratic program, we obtain the unique solution $x^* = (194.6750, 300)^T$. Since the gap function g_{D_*} for this problem at x^* is zero,

x^* is a local solution of (7), but, since $g_D(x^*) = 616$, this point is not a global solution to (7). Note that $u^* = (400, 300)^T$ is a global solution as

$$\begin{aligned} g_D(u^*) &= -\min_y\{\langle Au^* - \alpha, y\rangle + \varphi(y) : y \in D\} + \langle Au^* - \alpha, u^*\rangle + \varphi(u^*) \\ &= -\min_{0\le y_1\le 400}(Au^* - \alpha)_1 y_1 + \varphi_1(y_1) \\ &\quad -\min_{0\le y_2\le 300}(Au^* - \alpha)_2 y_2 + \varphi_2(y_2) + \langle Au^* - \alpha, u^*\rangle + \varphi(u^*) = 0. \end{aligned} \quad (8)$$

Now, we consider two practical models that can be formulated in the form of mixed variational inequality problem (P). Then, we present computational results and experiences for these models.

The Nash-Cournot oligopolistic market model is one of fundamental models in economics that has earned the attention of many authors (see, e.g., [1, 4, 5, 8, 12–14] and the references cited therein). In this model, it is assumed that there are N-firms producing a common homogeneous commodity. Each firm i has a strategy set $D_i \subset \mathbb{R}_+$ and a profit function f_i defined on the strategy set $D := D_1 \times \cdots \times D_N$ of the model. Let $x_i \in D_i$ be a corresponding production level of firm i. As usual, suppose that the profit function of firm i is given by

$$f_i(x) = p(\sigma)x_i - c_i(x_i) \ (i = 1, \ldots, N), \quad (9)$$

where $\sigma := \sum_{j=1}^N x_j$, $p(\sigma) := \alpha - \beta\sigma$ with $\beta > 0$, $\alpha > 0$ and, for every i, $c_i(x_i)$ is the cost for production x_i.

Actually, each firm seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept.

We recall that a point (strategy) $x^* = (x_1^*, \ldots, x_N^*)^T \in D$ is said to be a Nash equilibrium point of this Nash-Cournot oligopolistic market model if

$$f_i(x^*) \ge f_i(x^*[x_i]), \forall x_i \in D_i, \forall i,$$

where the vector $x^*[x_i]$ is obtained from x^* by replacing x_i^* with x_i.

To formulate the problem of finding a Nash equilibrium point in the form of mixed variational inequality (P), first we define the Nikaido-Isoda bifunction ϕ by taking

$$\begin{aligned} \phi(x, y) &= \sum_{i=1}^N \Big(f_i(x) - f_i(x[y_i])\Big) \\ &= \langle\bar{\alpha}, x - y\rangle + \beta\sum_{i=1}^N\left(\sum_{j\ne i}^N x_j\right)(y_i - x_i) + \beta\sum_{i=1}^N y_i^2 + \sum_{i=1}^N c_i(y_i) \\ &\quad -\beta\sum_{i=1}^N x_i^2 - \sum_{i=1}^N c_i(x_i) \\ &= \langle B_1 x - \bar{\alpha}, y - x\rangle + y^T B y + c(y) - x^T B x - c(x), \end{aligned}$$

where

$$B_1 = \begin{pmatrix} 0 & \beta & \ldots & \beta \\ \beta & 0 & \ldots & \beta \\ \cdot & \cdot & \cdot & \cdot \\ \beta & \beta & \ldots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \beta & 0 & \ldots & 0 \\ 0 & \beta & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \beta \end{pmatrix}, \quad c(x) = \sum_{i=1}^N c_i(x_i).$$

Thus,

$$\phi(x, y) = \langle F(x), y - x\rangle + \varphi(y) - \varphi(x),$$

where

$$F(x) = B_1 x - \bar{\alpha};\ \varphi(x) = x^T Bx + c(x) = \beta \sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} c_j(x_j).$$

So the problem of finding a Nash equilibrium point of this model can take the form of the mixed variational inequality (P). Note that this formulation was presented in [19].

In this case, the gap function takes the form

$$g_D(x) = -\min_{y\in D}\left\{\langle B_1 x - \bar{\alpha}, y - x\rangle + \sum_{i=1}^{N}(\beta y_i^2 + c_i(y_i) - \sum_{i=1}^{N}(\beta_i x_i^2 + c_i(x_i)\right\},$$

whereas the gap function of the model with convex envelope cost becomes

$$\bar{g}_D(x) = -\min_{y\in D}\left\{\langle B_1 x - \bar{\alpha}, y - x\rangle + \sum_{i=1}^{N}(\beta y_i^2 + \bar{c}_i(y_i) - \sum_{i=1}^{N}(\beta x_i^2 + \bar{c}_i(x_i)\right\}.$$

Note that in this model, since $D = D_1 \times \cdots \times D_N$, the convex envelope $\bar{c}_i$ of each function c_i on an interval is an affine function that can be computed by a closed form, moreover $\bar{c}(x_1, \ldots, x_N) = \sum_{i=1}^{N} \bar{c}_i(x_i)$.

The second example is the Bertrand model of oligopoly, where, as before, the firms produce a common homogenous commodity; however, in contrast to the Cournot model, here, each firm sets prices rather than the production quantity. So, in such a model, the demand is a function of price and the customers buy from firms with the lowest price. However, often this assumption is not realistic, since usually the products of the firms are not entirely interchangeable, and thus, some consumers may prefer one product to the other even it costs somewhat more.

Suppose that the quantity level x_i produced by firm i depends on the price p and is given by

$$x_i(p) = \gamma_i - \sigma_i p_i + \sum_{j\neq i}^{N} \lambda_{ij} p_j,\ i = 1, \ldots, N,$$

where $\gamma_i,\ \sigma_i > 0, \lambda_{ij} \geq 0$ if $(j \neq i)$. The condition $\sigma_i > 0$ means that the demand for firm i decreases as its price increases, while $\lambda_{ij} \geq 0.\ (i \neq j)$ means that the demand for firm i increases when other firms increase their price.

The profit function of firm i then is given as

$$f_i(p) := p_i x_i - \varphi_i(x_i),$$

where, following [2], we assume that the cost $\varphi_i(.)$ is a concave function of the production level and is given by

$$\varphi_i(x_i) = \nu_i x_i - d_i x_i^2 \text{ with } d_i \geq 0.$$

Then an elementary computation shows that the cost is a function of the price as

$$\varphi_i(p) = -d_i\sigma_i^2 p_i^2 + \sigma_i\left[2d_i\left(\gamma_i + \sum_{j\neq i}^{N}\lambda_{ij} p_j\right) - \nu_i\right] p_i + \nu_i\left(\gamma_i + \sum_{j\neq i}^{N}\lambda_{ij} p_j\right)$$
$$-d_i\left(\gamma_i + \sum_{j\neq i}^{N}\lambda_{ij} p_j\right)^2.$$

The profit function then takes the form

$$f_i(p) = \sigma_i(d_i\sigma_i - 1)p_i^2 + \left[\sigma_i \nu_i + \left(\gamma_i + \sum_{j\neq i}^{N} \lambda_{ij} p_j\right)(1 - 2d_i\sigma_i)\right] p_i$$
$$+ d_i \left(\gamma_i + \sum_{j\neq i}^{N} \lambda_{ij} p_j\right)^2 - \nu_i \left(\gamma_i + \sum_{j\neq i}^{N} \lambda_{ij} p_j\right).$$

Each firm i attempts to maximize its profit by choosing a corresponding price level on its strategy set D_i by solving the optimization problem

$$f_i(p) = \max_{y_i \in D_i} f_i(p[y_i]), \forall i = 1, \ldots, N,$$

where $p[y_i]$ is the vector obtained from p by replacing p_i with y_i.

By the same technique as in the Nash-Cournot model, the problem of finding a Nash equilibrium point of this Bertrand model can be formulated as a mixed variational inequality of the form

$$\text{Find } p \in D := D_1 \times \cdots \times D_N : \langle Gp + r, p - y\rangle + \varphi(y) - \varphi(p) \geq 0, \forall y \in D, \quad (10)$$

where

$$G = \begin{pmatrix} 0 & \lambda_{12}(1-2d_1\sigma_1) & \ldots & \lambda_{1N-1}(1-2d_1\sigma_1) & \lambda_{1N}(1-2d_1\sigma_1) \\ \lambda_{21}(1-2d_2\sigma_2) & 0 & \ldots & \lambda_{2N-1}(1-2d_2\sigma_2) & \lambda_{2N}(1-2d_2\sigma_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_{n1}(1-2d_N\sigma_N) & \ldots & \ldots & \lambda_{N,N-1}(1-2d_N\sigma_N) & 0 \end{pmatrix}$$

with

$$r^T = (r_1, \ldots, r_N), r_i = \sigma_i \nu_i + \gamma_i(1 - 2d_i\sigma_i),\ i = 1, \ldots, N, \varphi(y) = \sum_{i=1}^{N} \sigma_i(d_i\sigma_i - 1)y_i^2.$$

So the Bertrand model can be formulated as a mixed variational inequality (P). Note that the outputs in this model are prices rather the production level as in the Nash-Cournot model. Since $\sigma_i(d_i\sigma_i - 1)$, $i = 1, \ldots, N$ may be negative, the function $\varphi(.)$ may be concave. The concavity of φ implies $\sigma_i d_i \leq 1$ for all i, which gives a relationship between the reducing coefficients of the price and the cost. In some practical models, the price reducing coefficient is small. Note that, since the matrix G is in general, neither symmetric nor positive semidefinite (10) cannot be formulated in the form of a mathematical programming problem when φ is replaced by its convex envelope.

We now apply the proposed algorithm to solve the differentiated Nash-Cournot equilibrium model presented above by using Algorithm 2. The algorithm was implemented in MATLAB and executed on a PC Core 2Duo 2*2.0 GHz, RAM 2GB. We tested the program on different groups of problems, each of them contains ten problems of different sizes N and n, but having randomly generated input data. Namely, for each problem, the numbers α, β, μ_i $(i = n+1, \ldots, N)$ are randomly generated in the interval [20.30], [0.001, 0.005], and [10.20] respectively. We take the cost functions of the forms

$$c_j(x_j) = a_j x_j + \ln(1 + \gamma_j x_j),\ (j = 1, \ldots, n), \quad (11)$$
$$c_i(x_j) := \mu_i x_i,\ (i = n+1, \ldots, N), \quad (12)$$

where γ_j and a_j are randomly generated in [7, 15] and [2, 7] respectively. The strategy set of firm i is $D_i := [0, u_i]$ where each u_i is randomly generated in the interval [100.500].

The obtained results are reported in Table 1 below, where we use the following headings:

- N: number of the firms;
- n: number of the firms having concave (but not affine) cost;
- *Average time*: the average time (in second) needed to solve one problem;
- *Average iter*: the average numbers of iterations for one problem;
- n^*: number of the problems for which a global solution is obtained by evaluating the gap function at the currently best feasible point.

To test the algorithm for the Bertrand model, we take the concave quadratic function $\varphi_j(x_j) = \nu_j x_j - d_j x_j^2$ with $d_j > 0$ for every j, and we assume that the matrix G is symmetric, which implies that the mixed variational inequality subproblems can be formulated equivalently as quadratic (not necessarily convex) mathematical programs. The obtained computational results with random data are reported in Table 2. From the computational results reported in the tables, we can conclude the following:

For the Nash-Cournot, from Table 1, one can see that for the tested concave cost functions given as (11) the algorithm can solve models with a moderate number ($n \leq 40$) of concave cost functions, while the total number N may be much larger (two hundreds). The computational times for this model mainly spent to compute minimal points of one-dimensional DC

Table 1 For the Nash-Cournot model

N	n	Average time (s)	Average iter.	n^*
5	5	0.00	1	10
50	5	8.98	133	10
100	5	17.89	171	10
200	5	1.78	7	10
10	10	9.65	308	10
50	10	82.35	1141	10
100	10	47.05	445	10
200	10	41.06	203	10
20	20	127.15	2478	10
50	20	98.10	1231	10
100	20	105.00	914	10
200	20	440.88	2216	10
30	30	286.57	3754	10
50	30	246.44	2901	10
100	30	872.27	7193	10
200	30	750.72	3514	10
40	40	515.10	5944	10
50	40	1332.10	14820	9
100	40	646.53	5213	10
200	40	898.09	4169	9
100	100	Skip	–	–
200	100	Skip	–	–
200	200	Skip	–	–
300	200	Skip	–	–

Table 2 For the Bertrand model

N	Average time (s)	Average iter.
10	0.0004	1
20	0.00005	1
30	0.151	4
40	0.788	15
50	0.0468	2
60	0.172	4
70	85.0	868
80	99.1	926
90	523.0	2209
100	268.0	2062
120	836.0	5404
150	555.0	2913
200	253.0	825
300	300.0	74
400	6270.0	4004

functions on intervals of each edge of generated subboxes. The number of these subboxes is large when n is somewhat large. As we can see from Table 1, when $n \geq 100$, we had to skip the program as the number of the iterations exceeds two thousands.

For the Bertrand model with concave quadratic cost functions, the results reported in Table 2 show that the algorithm is quite efficient. The reason is that, for this case, evaluating the gap function at each currently obtained feasible point can be done by minimizing one-dimensional concave functions on an interval that is very easy to do by evaluating the function at the two end points of the interval. The most running times spent to globally solve quadratic programming subproblems.

4 Conclusion

The concave (resp. DC) mixed variational inequality can be considered as a development of the concave (resp. DC) minimization problem. We have proposed an algorithm for finding a global solution of these nonconvex mixed variational inequality problems on a box with separable cost functions. The algorithm uses the convex envelope of a concave function over a box to approximate the nonconvex original problem by convex ones. The convergence of the algorithm has been shown by employing an adaptive bisection procedure taking on the space of the concave variables. We have applied the algorithm to solve Nash-Cournot and Bertrand equilibrium models with concave cost. Some computational results on randomly generated data have been reported. An open question that would be interesting for further consideration is to find branch-and-bound and/or cutting plane algorithms for concave as well as DC mixed variational inequality problems over a polyhedral convex set.

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